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ON THE SOLUTION OF LINEAR EQUATIONS IN INFINITELY MANY VARIABLES

by

JOSEPH LEONARD WALSH

A Thesis Submitted for the Degree of
MASTER OF SCIENCE

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1.HISTORICAL INTRODUCTION.

In this thesis we shall consider systems of equations, of the general type

$$(1)\quad \begin{cases} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots = C_1, \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \cdots = C_2, \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + \cdots = C_3, \\ \cdots\cdots\cdots\cdots\cdots\cdots, \end{cases}$$

where the A_{ij} and C_i are given real[1] constants and the x_i are to be determined.

The first person to treat successfully such a system was Fourier, who in 1822 was led to a system of type (1) by attempting to obtain a solution of a certain differential equation in the form of a trigonomertic series. Fourier considered the auxiliary system

$$(2)\quad \begin{cases} A_{11}x_1^{(n)} + A_{12}x_2^{(n)} + \cdots + A_{1n}x_n^{(n)} = C_1, \\ A_{21}x_1^{(n)} + A_{22}x_2^{(n)} + \cdots + A_{2n}x_n^{(n)} = C_2, \\ \cdots\cdots\cdots\cdots, \\ A_{n1}x_1^{(n)} + A_{n2}x_2^{(n)} + \cdots + A_{nn}x_n^{(n)} = C_n, \end{cases}$$

which has the solution

$$x_k^{(n)} = \frac{\Delta_k^{(n)}}{\Delta^{(n)}}\,. \qquad \begin{cases} k = 1, 2, \cdots n, \\ n = 1, 2, 3, \cdots . \end{cases}$$

In his numerical case, he found that the limits

$$x_k = \lim_{n=\infty} x_k^{(n)}$$

actually furnished a solution of his original system (1). Fourier proved nothing about the general validity of this method for solving systems of the type (1); he was primarily interested in finding a solution for his own numerical case.

---

[1] In fact, the whole discussion holds almost verbatim for complex numbers also.

A method somewhat similar to that of Fourier was used by G.W.Hill[1] in 1877. Like Fourier, he was led to system[2] (1) while trying to solve a certain differential equation by means of a trigonometric series. Instead of considering the auxiliary systems (2) and the corresponding determinants, Hill considered directly the infinite determinant

$$\Delta = \begin{vmatrix} A_{11} & A_{12} & \\ A_{21} & A_{22} & \\ - & - & - \\ - & - & \end{vmatrix} = \lim_{n=\infty} \begin{vmatrix} A_{11} & A_{12} & & A_{1n} \\ A_{21} & A_{22} & - & A_{2n} \\ - & & & \\ A_{n1} & A_{n2} & & A_{nn} \end{vmatrix} .$$

Hill manipulated this determinant- a definite numerical expression in his case - precisely as one ordinarily manipulates a finite determinant. Like Fourier, he was simply interested in a numerical case and hence did not consider the general legitimacy of thus operating with infinite determinants. In fact, Hill did not even justify his own numerical results. But a few years later, in 1885, Poincaré took up the general study of infinite determinants and their use to solve systems of type (1). Poincaré showed that the formal work of Hill was justified. Later we shall consider more in detail the results of Poincaré as extended by von Koch.

Historically, then, operations with infinite determinants proved useful and valid in the first cases where they were performed. This immediately raises the question whether such operations are always valid - in particular, whether the method of Fourier will always lead to a solution of the system (1). We turn first to the answer of this question.

---

[1] "On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon", *Acta Mathematica*, t.8.

[2] Hill considered his system in a slightly different form, but it can be readily put into the form (1).

2.CONTRAST WITH FINITE SYSTEMS - Numerical Examples.

There are three principal properties of systems of n linear equations in n unknowns, when the determinant of the system is not zero:

a) there is always one solution,

b) there is only one solution,

c) this solution can always be found by the use of determinants.

It will be shown, by examples, that none of these properties is invariably true for infinite systems of type (1) with non-vanishing determinant. For simplicity, consider systems (1) such that $A_{ii}=1$ for every i, and $A_{ij}=0$ for every $i>j$. The determinants of these systems are unity, as are the determinants of the auxiliary systems (2) which are used in applying the method of Fourier.

a)From the system

$$\begin{aligned} x_1+x_2+x_3+\cdots &= \tfrac{1}{2}, \\ x_2+x_3+\cdots &= -\tfrac{1}{2}, \\ x_3+\cdots &= \tfrac{1}{2}, \\ \cdots &\cdots, \end{aligned}$$

it follows by subtracting the (k+1)-th equation from the k-th equation that if there is a solution it must be

$$x_k=(-1)^{k+1}, \quad (k=1,2,3, \quad).$$

When we substitute these values in the left hand members of our equations, we obtain series which diverge. However, if we use the word "solution" to include a set of values for which the series in the left hand members of the original equations do not converge but are summable to the corresponding right hand members, then the above values are a solution.

A formal solution may lead even to convergent series, whose sums are different from the corresponding right hand members of the original equations. For example, the system

$$\begin{aligned} x_1+x_2+x_3+\cdots &= m, \\ x_2+x_3+\cdots &= m, \\ x_3+\cdots &= m, \\ \cdots &\cdots, \end{aligned}$$

has the formal solution $x_k = c$ for every k. For these values, the series converge, but we have an actual solution when and only when $m = 0$.

Also, a formal solution may make the left hand members diverge with infinite sum, as in the system

$$\begin{aligned} x_1 + x_2 + x_3 + \dots &= 1, \\ x_2 + x_3 + \dots &= 2, \\ x_3 + \dots &= 3, \\ \dots \quad \dots &\dots . \end{aligned}$$

The formal solution is $x_k = -1$, but for these values the corresponding series diverge with infinite sum.

b) There exist systems with non-vanishing determinant which have an infinite number of solutions. For example, the system

$$\begin{aligned} x_1 - x_2 &= 0, \\ x_2 - x_3 &= 0, \\ x_3 - x_4 &= 0, \\ \dots \quad \dots &\dots , \end{aligned}$$

has not only the evident solution $x_k = 0$, but also the solution $x_k = n$ for every k, whatever the value of n.

c) There exist systems with non-vanishing determinant which have an infinite number of solutions, and yet no solution can be found by the determinantal method of Fourier. For consider the system

$$\begin{aligned} x_1 - x_2 &= 1, \\ x_2 - x_3 &= 1, \\ x_3 - x_4 &= 1, \\ \dots \quad \dots &\dots . \end{aligned}$$

The auxiliary systems (2) of Fourier have the solutions

$$\begin{aligned} &x_1^{(1)} = 1, \\ &x_2^{(2)} = 1, \quad x_1^{(2)} = 2, \\ &x_3^{(3)} = 1, \quad x_2^{(3)} = 2, \quad x_1^{(3)} = 3, \\ &\dots \quad \dots \quad . \end{aligned}$$

It follows that $\lim_{n\to\infty} X_k^{(n)} = \infty$ for every k. But clearly the system has an infinite number of solutions, since X_1 may be chosen arbitrarily.

These illustrations show, then, that the method of determinants is not always applicable. However, if the A_{ij} and the C_i of system (1) are suitably restricted, the method is applicable. A general application of this method was first developed by Poincaré and was later extended by von Koch.

3.STATEMENT OF THE RESULTS OF VON KOCH[1]

Consider the system

$$(1)\quad \begin{cases} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots = C_1, \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \cdots = C_2, \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + \cdots = C_3, \\ \cdots \end{cases}$$

under the following restrictions:

$$|C_i| \leqq C \quad (i = 1, 2, 3, \cdots), \quad \prod_{i=1}^{\infty} A_{ii} \text{ absolutely}$$

convergent, and $\sum_{\substack{i,j=1 \\ i \neq j}}^{\infty} |A_{ij}|$ convergent.

Place

$$\Delta = \lim_{n=\infty} \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & A_{2n} \\ & \cdots & & \\ A_{n1} & A_{n2} & & A_{nn} \end{vmatrix};$$

it can be shown that this limit always exists. Also place

$$\Delta_k = \lim_{n=\infty} \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1,k-1} & C_1 & A_{1,k+1} & & A_{1n} \\ & \cdots & & & & & & \\ A_{n,1} & A_{n,2} & & A_{n,k-1} & C_n & A_{n,k+1} & & A_{nn} \end{vmatrix};$$

---

[1] There is a bibliography at the end of this thesis.

this limit also must exist. Under the restrictions indicated, von Koch establishes the following results:

THEOREM Ia. If $\Delta \neq 0$, $x_k = \frac{\Delta_k}{\Delta}$ is a solution of (1); the x_k are limited, and this is the only solution for which the x_k are limited.

THEOREM Ib. If $\Delta = 0$, the homogeneous system corresponding to (1) always has a solution other than the evident solution $x_k = 0$. The system (1) itself has either no solution [in which case equations (1) are inconsistent] or has an infinite number of solutions.[1]

The use of infinite determinants is by no means the only method of solving equations of type (1). Next I shall use a direct method of successive approximations - a method similar to the Picard approximation process which is well known in connection with the theory of differential equations[2]. So far as I am aware, this method has never been previously applied - in either theory or practice - to solve systems of type (1).[3] The theory as developed below gives not only a number of new theorems, but also includes THEOREM Ia of von Koch. Moreover, the method of successive approximations is not only useful for the theory as developed below; it is a practical method for numerical computation.

---

[1] If the system (1) has more than one solution, the difference of any two solutions is a solution of the corresponding homogeneous system. Any multiple of the last solution is also a solution of the homogeneous system. Hence, the original system has an infinite number of solutions.

[2] See also the second footnote below.

[3] See, however, the Appendix.

4. METHOD OF SUCCESSIVE APPROXIMATIONS.

It will be convenient to write our equations in the form

$$(3)\quad \begin{cases} (1+a_{11})x_1 + a_{12}x_2 + a_{13}x_3 + \dots = c_1, \\ a_{21}x_1 + (1+a_{22})x_2 + a_{23}x_3 + \dots = c_2, \\ a_{31}x_1 + a_{32}x_2 + (1+a_{33})x_3 + \dots = c_3, \\ \dots\dots\dots . \end{cases}$$

The following theorem will now be established:.

THEOREM II. *Under the restrictions*

$$|c_i| \leq C, \quad \sum_{j=1}^{\infty} |a_{ij}| \leq P < 1, \quad (i = 1, 2, 3, \dots)$$

the system (3) *has always one solution and only one solution such that the* x_i *are bounded.*

For successive approximations[1] place[2]

$$x_k^{(1)} = c_k, \qquad (k = 1, 2, 3, \dots)$$

$$x_k^{(2)} = c_k - \left[a_{k1}x_1^{(1)} + a_{k2}x_2^{(1)} + \dots\right],$$

and in general, for $j > 0$,

$$(4)\quad x_k^{(j+1)} = c_k - \left[a_{k1}x_1^{(j)} + a_{k2}x_2^{(j)} + \dots\right].$$

Then

$$(5)\quad \begin{cases} x_k^{(1)} = c_k, \\ x_k^{(2)} - x_k^{(1)} = -\left[a_{k1}x_1^{(1)} + a_{k2}x_2^{(1)} + a_{k3}x_3^{(1)} + \dots\right], \\ x_k^{(3)} - x_k^{(2)} = -\left[a_{k1}(x_1^{(2)} - x_1^{(1)}) + a_{k2}(x_2^{(2)} - x_2^{(1)}) + a_{k3}(x_3^{(2)} - x_3^{(1)}) + \dots\right], \\ x_k^{(4)} - x_k^{(3)} = -\left[a_{k1}(x_1^{(3)} - x_1^{(2)}) + a_{k2}(x_2^{(3)} - x_2^{(2)}) + a_{k3}(x_3^{(3)} - x_3^{(2)}) + \dots\right], \\ \dots\dots\dots , \end{cases}$$

---

[1] The method is analogous to, and was suggested by the Picard process. See Lalesco, Theorie des Equations Intègrales, p.7.

[2] Any other bounded set would serve for the first set of approximations.

and therefore

$$(6)\quad \begin{cases} |X_k^{(1)}| \leqq C, \\ |X_k^{(2)} - X_k^{(1)}| \leqq C\rho, \\ |X_k^{(3)} - X_k^{(2)}| \leqq C\rho^2, \\ \quad - \quad - \quad - \quad, \\ |X_k^{(\nu)} - X_k^{(\nu-1)}| \leqq C\rho^{\nu-1}, \\ \qquad . \end{cases}$$

It follows that we can set

$$(7)\qquad X_k = X_k^{(1)} + (X_k^{(2)} - X_k^{(1)}) + (X_k^{(3)} - X_k^{(2)}) + \cdots ,$$

since the series on the right converges.

The X_k, as thus defined, are a solution of equations (3). For adding equations (5), on the right we have a double series that is absolutely convergent. When we sum ~~the sum~~ the double series by columns, we find

$$X_k = c_k - \left[a_{k1}X_1 + a_{k2}X_2 + \cdots \right],$$

so the X_k are a solution of (3).

It follows from equations (6) and (7) that

$$|X_k| \leqq C + C\rho + C\rho^2 + \cdots ,$$

and therefore the X_k are limited. It will next be shown that there is only one solution of (3) for which this is true. If X_k' and X_k'' denote two such solutions of (3), then $\bar{X}_k = X_k' - X_k''$ is limited, and $\bar{X}_k$ is a solution of the homogeneous system corresponding to (3):

$$(8)\quad \begin{cases} (1+a_{11})\bar{X}_1 + a_{12}\bar{X}_2 + a_{13}\bar{X}_3 + \cdots = 0, \\ a_{21}\bar{X}_1 + (1+a_{22})\bar{X}_2 + a_{23}\bar{X}_3 + \cdots = 0, \\ a_{31}\bar{X}_1 + a_{32}\bar{X}_2 + (1+a_{33})\bar{X}_3 + \cdots = 0, \\ - \quad - \quad - \quad - \quad - \quad - \quad - \quad . \end{cases} \qquad |\bar{X}_k| \leqq \bar{X}.$$

Carry through the approximation process for this system:

$$(9)\quad \begin{cases} X_k^{(1)} = \bar{X}_k, \\ X_k^{(2)} = -\left[a_{k1}X_1^{(1)} + a_{k2}X_2^{(1)} + \cdots \right], \\ \quad - \quad - \quad - \quad - \\ X_k^{(\nu+1)} = -\left[a_{k1}X_1^{(\nu)} + a_{k2}X_2^{(\nu)} + \cdots \right], \end{cases}$$

Then

$$|X_k'| \leq \overline{X} \,,$$
$$|X_k^{(2)}| \leq \overline{X}\rho \,,$$
$$|X_k^{(3)}| \leq \overline{X}\rho^2 \,,$$
$$- \; - \; - \; .$$

Since $\rho < 1$, $\lim_{i=\infty} X_k^{(i)} = 0$. But from (8) and (9) there results

$$\overline{X}_k = X_k^{(1)} = X_k^{(2)} = \qquad .$$

Hence $\overline{X}_k = 0$, and therefore the two solutions X_k' and X_k'' are identical, which we were to prove.

It should be noticed that if we omit from the hypothesis the restriction $\rho < 1$ or even if we allow $\rho = 1$, the theorem would be untrue. In fact, it is possible to have in this case

a) no solution of the system. This is apparent from the inconsistent system

$$\begin{array}{llll} X_1 + X_2 & & & = c_1 \,, \\ X_1 + X_2 & & & = c_2 \,, \\ & X_3 & & = c_3 \,, \\ & & X_4 & = c_4 \,, \\ & & - \; - \; - & \; . \end{array}$$

$$c_1 \neq c_2 \,, \; |c_k| \leq C \quad (k = 1, 2, \qquad).$$

b) an infinite number of solutions such that the X_k are bounded; this is illustrated by the system

$$\begin{array}{llll} X_1 + X_2 & & & = c_1 \,, \\ X_1 + X_2 & & & = c_2 \,, \\ & X_3 & & = c_3 \,, \\ & & X_4 & = c_4 \,, \\ & & - & \; . \end{array}$$

$$c_1 = c_2 \,, \; |c_k| \leq C \quad (k = 1, 2, \qquad).$$

c) an infinite number of solutions, but no solution for which the X_k are bounded. This is obviously true of the system

$$
\begin{aligned}
x_1 - x_2 \qquad\qquad &= -1, \\
x_2 - x_3 \qquad &= -1, \\
x_3 - x_4 &= -1, \\
- \qquad\quad &.
\end{aligned}
$$

However, the fact that the theorem is not true for $\rho = 1$ does not mean that the conclusion necessarily fails nor that the approximation process can never be used in this case. For although we cannot use the approximation process in a) and c) as in both cases the series for x_k diverges with infinite sum for at least one value of k, we can use the process to solve the system

$$
\begin{aligned}
x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{4}x_3 + \tfrac{1}{8}x_4 + \cdots &= 2, \\
x_2 + \tfrac{1}{2}x_3 + \tfrac{1}{4}x_4 + \quad &= 2, \\
x_3 + \tfrac{1}{2}x_4 + \cdot \quad &= 2, \\
- - - - \qquad &.
\end{aligned}
$$

Though the series

$$x_k^{(1)} + (x_k^{(2)} - x_k^{(1)}) + (x_k^{(3)} - x_k^{(2)}) + \cdots$$

all diverge, they are summable to the values $x_k = 1$, and this set of values actually is a solution - indeed the only solution - of the original system.

It is seen, then, that for a general theorem some such restriction as $\rho \leqq 1$ is necessary. However, by specializing the form of the equations, this restriction can be lightened to a certain extent. This is done in

THEOREM III. If the system

$$
(10)\quad \left\{
\begin{aligned}
x_1 + a_{12}x_2 + a_{13}x_3 + \qquad &= c_1, \\
x_2 + a_{23}x_3 + \cdots &= c_2, \\
x_3 + \quad \cdots &= c_3, \\
- - - - \qquad &,
\end{aligned}
\right.
$$

satisfies the hypotheses

$$|c_i| \leqq C, \quad \sum_{j=i+1}^{\infty} |a_{ij}| \ \underline{\text{convergent}}, \quad (i = 1, 2, 3, \cdots)$$

$$\sum_{j=i+1}^{\infty} |u_{ij}| \leqq \rho < 1 \quad \textit{for every } i > N,$$

then the system (10) *has one solution and only one solution for which the* X_k *are bounded.*

This theorem can be proved by using the approximation process used in the proof of THEOREM II.[1] The two proofs are so much alike[2] that we omit the proof of THEOREM III.

THEOREMS II and III surely include cases not included in THEOREM Ia of von Koch. For example, in (10) take

$$a_{i,i+k} = a_{j,j+k} \neq 0. \quad (i,j,k = 1,2,\cdots)$$

The series $\sum_{i=1,\, j \geq i+1}^{\infty} |u_{ij}|$ diverges. On the other hand, we shall show later that THEOREM Ia can be included in THEOREM III. The method of proof is to start with the system (1), transform into the type (10), and to show that the transformed system satisfies the hypotheses of THEOREM III. It so frequently occurs that equations need to be transformed into the form (10) that it is worth while to devote a section to this transformation.

---

[1] Or, we can omit the first N equations and apply THEOREM II to the remainder. Then to find the first N unknowns we have merely to solve a system of N equations in N unknowns; the determinant of this finite system does not vanish.

[2] Perhaps the easiest way to prove THEOREM III by using the approximation process is to notice the truth of the theorem for N = 0, and then to demonstrate it for N = 1, 2, ..., thus making a proof by induction.

5. TRANSFORMATION OF SYSTEMS INTO SPECIAL FORM.

A very large class of systems of equations of the general type

$$(1)\quad \begin{cases} A_{11}x_1 + A_{12}x_2 + \quad = c_1\,, \\ A_{21}x_1 + A_{22}x_2 + \quad \cdot \quad = c_2\,, \\ \qquad\qquad\qquad\qquad\qquad , \end{cases}$$

can be transformed into equivalent systems of the special form

$$(11)\quad \begin{cases} x_1 + a_{12}x_2 + a_{13}x_3 + \cdot \qquad = \gamma_1\,, \\ \qquad\quad x_2 + a_{23}x_3 + \qquad = \gamma_2\,, \\ \qquad\qquad\qquad x_3 + \qquad = \gamma_3\,, \\ \qquad\qquad\qquad - \quad - \qquad . \end{cases}$$

Roughly speaking, to effect this transformation we leave unchanged the first equation of (1); use this first equation to eliminate x_1 from the second equation; use the first and second equations to eliminate x_1 and x_2 from the third equation, and so on. This leads to the system

$$(12)\quad \begin{cases} b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + \qquad = \,_1\,, \\ \qquad\quad b_{22}x_2 + b_{23}x_3 + \cdot \qquad = \quad _2\,, \\ \qquad\qquad\qquad b_{33}x_3 + \cdot \qquad = \quad ,\,, \\ \qquad\qquad\qquad - \quad - \quad - \quad , \end{cases}$$

which is readily put into the form (11).

To be more explicit, this transformation from (1) to (12) can always be accomplished if we suppose

$$\Delta^{(n)} = \begin{vmatrix} A_{11} & A_{12} & & A_{1n} \\ & & - & \\ A_{n1} & A_{n2} & & A_{nn} \end{vmatrix} \neq 0. \quad (n = 1, 2, - - -)$$

For in this case we have merely to place[1]

$$\begin{cases} b_{1k} = A_{1k}\,, \quad {}_1 = c_1\,, \end{cases}$$

---

[1] See Riesz, Les Systèmes d'Equations Linéaires a une infinité d'inconnues, p. 11.

$$
(13)\quad \begin{cases}
t_{nk} = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1,n-1} & A_{1k} \\ & & \cdots & & \\ A_{n1} & A_{n2} & \cdots & A_{n,n-1} & A_{nk} \end{vmatrix}, \\[2em]
\gamma_n = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1,n-1} & C_1 \\ \cdots & \cdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{n,n-1} & C_n \end{vmatrix}.
\end{cases}
$$

Presently it will be shown that the transformation from (1) to (12) [and hence to (11)] can be made if there exists an N such that $\Delta^{(n)} \neq 0$ for $n \geq N$. For this purpose we need the

LEMMA. *In any non-vanishing determinant,*

$$
\Delta^{(k)} = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & & \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{vmatrix},
$$

the rows can be arranged so that no minor of the form

$$
\Delta^{(i)} = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1i} \\ A_{21} & A_{22} & \cdots & A_{2i} \\ \cdots & & \cdots & \\ A_{i1} & A_{i2} & \cdots & A_{ii} \end{vmatrix} \qquad (i = 1, 2, \cdots, k)
$$

will vanish.

Surely $A_{i1} \neq 0$ for some value of i; otherwise we should have $\Delta^{(k)} = 0$. Hence by a change (if necessary) in the order of the rows, we can make $A_{11} \neq 0$. Let us suppose that $\Delta^{(1)} = A_{11} \neq 0$, $\Delta^{(2)} \neq 0$, $\Delta^{(n)} \neq 0$, $\Delta^{(n+1)} = 0$, and that by no further change in the order of the rows succeeding the n-th can we make $\Delta^{(n+1)} \neq 0$. Then each of the sets of constants

$$
A_{m1}, A_{m,2}, \cdots A_{m,n+1}, \quad (m = n+1, n+2, \cdots, k)
$$

is linearly dependent on the sets of constants

$$
A_{m1}, A_{m2}, \cdots A_{m,n+1}. \quad (m = 1, 2, \cdots, n)
$$

Hence we can write $\Delta^{(k)}$ in the form

$$\Delta^{(k)} = \begin{vmatrix} A_{11} & A_{12} & & A_{1\,n+1} & A_{1\,n+2} & & A_{1k} \\ A_{21} & A_{22} & & A_{2\,n+1} & A_{2,n+2} & & A_{2k} \\ & - & - & & - & - & \\ A_{n1} & A_{n2} & - & A_{n,n+1} & A_{n\,n+2} & & A_{nk} \\ 0 & 0 & - & 0 & A'_{n+1,n+2} & & A'_{n+1,k} \\ 0 & 0 & - & 0 & A'_{n+2,n+2} & - & A'_{n+2,k} \\ & \cdot & - & \cdot & & & \cdot \\ 0 & 0 & - & 0 & A'_{k,n+2} & & A'_{kk} \end{vmatrix}. \qquad (k \geq n+1)$$

Therefore $\Delta^{(k)} = 0$, contrary to our hypothesis. This completes the proof of the lemma.

If the system (1) ,then, is such that there exists an N so that $\Delta^{(n)} \neq 0$ whenever $n \geq N$, the first N equations can be arranged to make $\Delta^{(k)} \neq 0$ for $k = 1, 2, \quad N$. This rearrangement does not change the values of $\Delta^{(n)}$ for $n > N$. Hence we can arrange the equations so that $\Delta^{(n)} \neq 0$ for every value of n .

In particular, if we are dealing with the infinite determinant which appears in THEOREM Ia, $\Delta \neq 0$, it is surely true that there exists N such that $\Delta^{(n)} \neq 0$ for $n \geq N$, and hence the equations can be arranged so that $\Delta^{(n)} \neq 0$ for every n .

6. PROOF OF SOME RESULTS OF VON KOCH.

Let us consider the system

$$
(1) \quad \left| \begin{array}{ll}
A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \qquad & = C_1, \\
A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \qquad & = C_2, \\
A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + \qquad & = C_3, \\
 & \quad ,
\end{array} \right.
$$

under the hypotheses of von Koch:

$$|C_i| \leq C \quad (i = 1, 2, \quad), \quad \prod_{i=1}^{\infty} A_{ii} \text{ absolutely convergent,}$$

$$\text{and} \quad \sum_{\substack{i,j=1 \\ i \neq j}}^{\infty} |A_{ij}| \text{ convergent.}$$

We shall consider only the case treated in THEOREM Ia, namely, that in which $\Delta \neq 0$. Further, it will be supposed that the equations have been arranged so that $\Delta^{(n)} \neq 0$ $(n = 1, 2, \quad)$. Rearrangement of any system (1) to satisfy this last restriction would not disturb the convergence of the double series or infinite product.

Replace system (1) by an equivalent system of type (12), using formulas (13). If we set

$$A_k = |1 - A_{kk}| + \sum_{\substack{i=1 \\ i \neq k}}^{\infty} |A_{ik}| ,$$

then $\sum_{k=1}^{\infty} A_k$ converges. Hence $\prod_{k=1}^{\infty} (1 + A_k)$ also converges.

Using the relation $|C_k| \leq C$, we have

$$
(14) \quad (\text{abs. val. of}) \left| \begin{array}{llll}
A_{11} & A_{12} & \quad A_{1n-1} & C_1 \\
 & \quad - - & & \\
A_{n1} & A_{n2} & \quad A_{nn-1} & C_n
\end{array} \right| \leq C \cdot \prod_{k=1}^{\infty} (1 + A_k).
$$

For to every term on the left there corresponds a positive term on the right which is not smaller than that term. Hence the β_n are limited.

Next we shall prove that the series of coefficients in the $\underline{i}$-th row of (12) converges absolutely, and that the sum of the absolute values of the terms can be made as small as desired for every $i > I$, where I is properly chosen. From (13), using the method used in (14), we see that for $n < k$,

$$|b_{nk}| \leq \Pi \cdot \left[|A_{1k}| + |A_{2k}| + \qquad + |A_{nk}|\right].$$

Therefore

$$|b_{n,n+1}| \leq \Pi \cdot \left[|A_{1,n+1}| + |A_{2,n+1}| + \qquad + |A_{n,n+1}|\right],$$

$$|b_{n,n+2}| \leq \Pi \cdot \left[|A_{1,n+2}| + |A_{2,n+2}| + \qquad + |A_{n,n+2}|\right],$$

Hence, if we put

$$B_n = |b_{n,n+1}| + |b_{n,n+2}| + \cdots \qquad ,$$

we get

$$\begin{aligned} B_n \leq \Pi \cdot \Big[& |A_{1,n+1}| + |A_{1,n+2}| + \\ & + |A_{2,n+1}| + |A_{2,n+2}| + \\ & \\ & + |A_{n,n+1}| + |A_{n,n+2}| + \qquad \Big]. \end{aligned}$$

Then for every value of $\underline{n}$, the series $\underline{B_n}$ converges. From the convergence of

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{\infty} |A_{ij}|,$$

it follows that B_n can be made arbitrarily small simply by taking $n > N$, where N is properly chosen.

Then we have transformed the system (1) into a system (12), such that the z_i are limited, the series of the absolute values of the coefficients in the n-th row converges for every n, and its sum can be made as small as desired by taking $n > N$. None of the b_{nn} is zero; for $n = \infty$ they approach a limit different from zero. It follows that when system (12) is put into the form (11) that the hypotheses of THEOREM III are satisfied. Hence system (1), under the hypotheses given, has been transformed so as to be included in THEOREM III. Thereby has been proved the existential part of von Koch's THEOREM Ia.

7. FURTHER RESULTS- USING SUCCESSIVE APPROXIMATIONS.

We shall now restrict ourselves to equations of the form[1]

$$(15)\begin{cases} x_1 + a_{12}x_2 + a_{13}x_3 + \cdots = c_1, \\ \qquad x_2 + a_{23}x_3 + \cdots = c_2, \\ \qquad\qquad x_3 + \cdots = c_3, \\ \qquad\qquad\qquad \cdots\cdots, \end{cases}$$

and we shall prove a number of theorems about such systems.

Theorem IV a. Under the hypothesis

$$|c_k| \leqq MC^k, \quad \sum_{j=i+1}^{\infty} |a_{ij}| \leqq P \; (2), \quad C \leqq 1, \quad C < \frac{1}{P},$$

system (15) has one solution and only one solution for which $|x_k| \leqq u\gamma^k$, $\gamma \leqq 1$, $\gamma < \frac{1}{P}$. Here M, C, P, u, and γ denote positive constants.

Place

$$(16)\begin{cases} x_k^{(1)} = c_k, \\ x_k^{(2)} = c_k - [a_{k,k+1}x_{k+1}^{(1)} + a_{k,k+2}x_{k+2}^{(1)} + \cdots], \\ \cdots\cdots \\ x_k^{(i+1)} = c_k - [a_{k,k+1}x_{k+1}^{(i)} + a_{k,k+2}x_{k+2}^{(i)} + \cdots], \end{cases}$$

Then

$$(17)\begin{cases} x_k^{(1)} = c_k, \\ x_k^{(2)} - x_k^{(1)} = -[a_{k,k+1}x_{k+1}^{(1)} + a_{k,k+2}x_{k+2}^{(1)} + \cdots], \\ x_k^{(3)} - x_k^{(2)} = -[a_{k,k+1}(x_{k+1}^{(2)} - x_{k+1}^{(1)}) + a_{k,k+2}(x_{k+2}^{(2)} - x_{k+2}^{(1)}) + \cdots], \\ \cdots\cdots, \end{cases}$$

---

[1] Cf. §4.

(2) It is sufficient if this is true merely for $i > I$; cf. note 1 under Theorem III. Similarly in the other theorems.

and hence

$$|x_k^{(1)}| \leqq M C^k ,$$

$$|x_k^{(2)} - x_k^{(1)}| \leqq P M C^{k+1} ,$$

$$|x_k^{(3)} - x_k^{(2)}| \leqq P^2 M C^{k+2} ,$$

$$\cdot$$

Accordingly

$$x_k = x_k^{(1)} + (x_k^{(2)} - x_k^{(1)}) + (x_k^{(3)} - x_k^{(2)}) + \qquad (18)$$

$$\ll M C^k + P M C^{k+1} + P^2 M C^{k+2} + \cdots ,$$

which converges since $C < \frac{1}{P}$.

As defined by (18), the x_k actually form a solution of the original equations, for adding all the equations of (17), we have

$$x_k = c_k - [a_{k\,k+1} x_{k+1} + a_{k,k+2} x_{k+2} + \quad] .$$

This formal work is justifiable, as the double series we get from (17) converges absolutely.

It follows from (18) that

$$|x_k| \leqq M \frac{C^k}{1 - PC} .$$

Hence it is surely true that there exist numbers u and γ such ~~that~~ $|x_k| \leqq u \gamma^k$, $\gamma \leqq 1$, $\gamma < \frac{1}{P}$. That under this restriction the solution is unique, is readily seen

by a process exactly similar to that already used in Theorem II[1]. We omit the proof.

Theorem IV b. Under the restrictions

$$|C_k| \leqq MC^k \quad , \quad |a_{i,i+1}|^p + |a_{i,i+2}|^p + \qquad \leqq Q^p ,$$

$$C < \frac{1}{(1+Q^{\frac{p}{p-1}})^{\frac{p-1}{p}}} \qquad (p>1) ,$$

system (15) has one solution and only one solution such that

$$|x_k| \leqq u\gamma^k \quad , \quad \gamma < \frac{1}{(1+Q^{\frac{p}{p-1}})^{\frac{p-1}{p}}} .$$

Here M, C, Q, u, and γ denote positive constants.

We shall use the inequality[2]

$$(17) \qquad \left|\sum_{k=1}^{\infty} a_k b_k\right|^p \leqq \left(\sum_{k=1}^{\infty}|a_k|^p\right)\left(\sum_{k=1}^{\infty}|b_k|^{\frac{p}{p-1}}\right)^{p-1} .$$

When we use the approximation process given by (16), we find by (17) that

---

[1] Theorem II is the present theorem for $p<1$, $C=1$, and a more general form of the equations.

[2] See Riesz, l.c. p. 45.

$$|x_k^{1}| \leq MC^k,$$

$$|x_1^{(2)}-x_1^{(1)}|^p \leq Q^p\left[M^{\frac{1}{p-1}}C^{2\frac{p}{p-1}} + M^{\frac{p}{p-1}}C^{3\frac{p}{p-1}} + \quad\right]^{p-1}$$

$$= Q^p M^p\left[\frac{C^{2\frac{p}{p-1}}}{1-C^{\frac{p}{p-1}}}\right]^{p-1},$$

$$|x_k^{(2)}-x_k^{(1)}|^p \leq Q^p M^p\left[\frac{C^{(k+1)\frac{p}{p-1}}}{1-C^{\frac{p}{p-1}}}\right]^{p-1},$$

$$|x_1^{(3)}-x_1^{(2)}|^p \leq Q^{2p} M^p\left[\frac{C^{3\frac{p}{p-1}}}{1-C^{\frac{p}{p-1}}} + \frac{C^{4\frac{p}{p-1}}}{1-C^{\frac{p}{p-1}}} + \quad\right]^{p-1}$$

$$= Q^{2p} M^p\left[\frac{C^{3\frac{p}{p-1}}}{(1-C^{\frac{p}{p-1}})^2}\right]^{p-1},$$

$$|x_k^{(3)}-x_k^{(2)}|^p \leq Q^{2p} M^p\left[\frac{C^{(k+2)\frac{p}{p-1}}}{(1-C^{\frac{p}{p-1}})^2}\right]^{p-1}.$$

Then

$$x_k = x_k^{(1)} + (x_k^{(2)}-x_k^{(1)}) + (x_k^{(3)}-x_k^{(2)}) +$$

$$\ll MC^k + QM\left[\frac{C^{(k+1)\frac{p}{p-1}}}{1-C^{\frac{p}{p-1}}}\right]^{\frac{p-1}{p}} + Q^2M\left[\frac{C^{(k+2)\frac{p}{p-1}}}{(1-C^{\frac{p}{p-1}})^2}\right]^{\frac{p-1}{p}} +$$

$$= MC^k\left\{1 + Q\left[\frac{C^{\frac{p}{p-1}}}{1-C^{\frac{p}{p-1}}}\right]^{\frac{p-1}{p}} + Q^2\left[\frac{C^{\frac{p}{p-1}}}{1-C^{\frac{p}{p-1}}}\right]^{2\frac{p-1}{p}} + \right\}.$$

The series converges, for

$$C < \frac{1}{\left[1+Q^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}}}.$$

It follows from equations (17) that the x_k, as defined above, are a solution of (15). Moreover,

$$|x_k| \leqq \frac{MC^k}{\left[1-Q\,\dfrac{C^{\frac{p}{p-1}}}{1-C^{\frac{p}{p-1}}}\right]^{\frac{p-1}{p}}},$$

so there exist numbers u and γ such that

$$|x_k| \leqq \gamma^k u \qquad \gamma < \frac{1}{\left[1+Q^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}}}.$$

Under these restrictions it is readily shown that the solution is unique.[1]

---

[1] Theorem IV b was [illegible] in the case $p=2$ by a different method. See Proc. Camb. Cong. Math. I, 354.

Theorem IV c. Under the restrictions

$$|u_{ik}| \leqq NT^{k-i}, \quad |c_k| \leqq MC^k, \quad TC < \frac{1}{1+N},$$

system (15) has one solution and only one solution for which

$$|x_k| \leqq u\gamma^k, \quad T\gamma < \frac{1}{1+N}.$$

We use N, T, M, C, u, and γ to denote positive constants.

The approximation formulas (16) lead to (17), from which we get

$$|x_k^{(1)}| \leqq MC^k,$$

$$|x_k^{(2)} - x_k^{(1)}| \leqq MN\left[TC^{k+1} + T^2C^{k+2} + \quad\right]$$

$$= \frac{MNTC^{k+1}}{1-TC},$$

$$|x_k^{(3)} - x_k^{(2)}| \leqq N\left[\frac{MNTC^{k+2}}{1-TC}T + \frac{MNTC^{k+3}}{1-TC}T^2 + \cdot\right]$$

$$= \frac{MN^2T^2C^{k+2}}{1-TC},$$

It follows that

$$x_k = x_k^{(1)} + (x_k^{(2)} - x_1^{(1)}) + (x_k^{(3)} - x_k^{(2)}) +$$

$$\ll MC^k + \frac{MNTC^{k+1}}{1-TC} + \frac{MN^2T^2C^{k+2}}{(1-TC)^2} + \cdots.$$

The last series converges, for $TC < \frac{1}{1+N}$.

We have

$$|x_k| \leqq \frac{MC^k}{1-\frac{NTC}{1-TC}},$$

so the x_k just determined are such that there exist numbers u and γ for which

$$|x_k| \leqq u\gamma^k \quad \text{where } T\gamma < \frac{1}{1+N}.$$

Under this last restriction, the x_k just found are easily shown to be unique.

We note that if in particular we set $T=1$, our restrictions become

$$|a_{ik}| \leqq N, \quad |c_k| \leqq MC^k, \quad C < \frac{1}{1+N}.$$

As a final illustration of this approximation process, we shall prove

Theorem V. If for the system

$$(20)\quad \left\{\begin{array}{llll} (1+\alpha_{11})x_1 + & \alpha_{12}x_2 + & \alpha_{13}x_3 + \cdots & = c_1, \\ \alpha_{21}x_1 + & (1+\alpha_{22})x_2 + & \alpha_{23}x_3 + \cdots & = c_2, \\ \alpha_{31}x_1 + & \alpha_{32}x_2 + & (1+\alpha_{33})x_3 + \cdots & = c_3, \\ \cdots & & & \cdots \end{array}\right.$$

the series

$$\gamma^2 \equiv \sum_{j=1}^{\infty} |c_j|^2, \quad \rho_i^2 \equiv \sum_{j=1}^{\infty} |\alpha_{ij}|^2, \quad \rho^2 \equiv \sum_{i=1}^{\infty} \rho_i^2$$

converge, and if $\rho^2 < 1$, then (20) has one solution and only one solution for which $\sum_{i=1}^{\infty} |x_i|^2$ converges.

From the inequality (19) for $p = 2$, namely,

$$\left|\sum_{k=1}^{\infty} a_k b_k\right|^2 \leqq \left(\sum_{k=1}^{\infty} |a_k|^2\right)\left(\sum_{k=1}^{\infty} |b_k|^2\right),$$

it follows that

$$\sum_{k=1}^{\infty} |x'_k|^2 \equiv \gamma^2,$$

$$|x_k^{(2)} - x'_k|^2 \leqq \rho_k^2 \gamma^2,$$

$$\sum_{k=1}^{\infty} |x_k^{(2)} - x'_k|^2 \leqq \rho^2 \gamma^2,$$

$$|x_k^{(3)} - x_k^{(2)}|^2 \leqq \rho_k^2 \rho^2 \gamma^2 ,$$

$$\sum_{k=1}^{\infty} |x_k^{(3)} - x_k^{(2)}|^2 \leqq \rho^4 \gamma^2 ,$$

Then

$$x_k = x_k^{(1)} + (x_k^{(2)} - x_k^{(1)}) + (x_k^{(3)} - x_k^{(2)}) + \cdots$$

$$\ll |c_k| + \rho_k \gamma + \rho_k \rho \gamma + \rho_k \rho^2 \gamma + \cdots .$$

$$|x_k|^2 \leqq 2 \{ |c_k|^2 + \rho_k^2 (\gamma + \rho\gamma + \rho^2\gamma + \cdots)^2 \} ,$$

$$\sum_{k=1}^{\infty} |x_k|^2 \leqq 2 \{ \gamma^2 + \rho^2 (\gamma + \rho\gamma + \rho^2\gamma + \cdots)^2 \} .$$

Hence $\sum_{k=1}^{\infty} |x_k|^2$ converges.

It is now easily shown that there cannot be two solutions x'_k and x''_k for which $\sum_{k=1}^{\infty} |x'_k|^2$ and $\sum_{k=1}^{\infty} |x''_k|^2$ converge. If there were two solutions, $\bar{x}_k = x'_k - x''_k$ would be a solution of the homogeneous system corresponding to (26). We carry out the approximation process, placing

$$\lambda_k^{(1)} = \bar{\lambda}_k,$$

$$\lambda_k^{(2)} = -(\alpha_{k1}\lambda_1^{(1)} + \alpha_{k2}\lambda_2^{(1)} + \cdots),$$

$$\lambda_k^{(3)} = -(\alpha_{k1}\lambda_1^{(2)} + \alpha_{k2}\lambda_2^{(2)} + \cdots),$$

$$\cdots\cdots$$

If we set $\sum_{k=1}^{\infty} |\bar{\lambda}_k|^2 = \sigma^2$, we find

$$\sum_{k=1}^{\infty} |\lambda_k^{(1)}|^2 = \sigma^2,$$

$$|\lambda_k^{(2)}|^2 \leq \rho_k^2 \sigma^2,$$

$$\sum_{k=1}^{\infty} |\lambda_k^{(2)}|^2 \leq \rho^2 \sigma^2,$$

$$|\lambda_k^{(3)}|^2 \leq \rho_k^2 \rho^2 \sigma^2,$$

$$\sum_{k=1}^{\infty} |\lambda_k^{(3)}|^2 \leq \rho^4 \sigma^2,$$

$$\cdots\cdots$$

Since $\rho < 1$, $\lim_{i=\infty} |\lambda_k^{(i)}| = 0$. But ~~by (20)~~ $\bar{\lambda}_k = \lambda_k^{(1)} = \lambda_k^{(i)}$ $(i = 1, 2, 3, \cdots)$. Hence $\bar{\lambda}_k = 0$.

If we replace (20) by the system

$$(21)\quad \begin{cases} x_1 + a_{12}x_2 + a_{13}x_3 + \cdots = c_1, \\ \qquad\quad x_2 + a_{23}x_3 + \cdots = c_2, \\ \qquad\qquad\qquad x_3 + \cdots = c_3, \\ \qquad\qquad\qquad \cdots\cdots, \end{cases}$$

where $\sum_{k=1}^{\infty} |c_k|^2$ converges, it is sufficient to assume $\sum_{j=i+1}^{\infty} |a_{ij}|^2$ convergent for every i, and

$$\sum_{i=I+1}^{\infty} \sum_{j=i+1}^{\infty} |a_{ij}|^2 \} < 1$$

for some I. For we can suppress the first I equations, carry out the approximation process on the others, and then solve the first I equations for $\lambda_1, \lambda_2, \ldots \lambda_I$.(1)

It may be noticed that the restrictions placed on (21) will surely be satisfied if

$$\sum_{\substack{i,j=1 \\ i<j}}^{\infty} |a_{ij}|^2$$

is convergent.

---

[1] In fact, it is not sufficient to show that the original approximation process is applicable to (21) directly.

8. SUCCESSIVE APPROXIMATIONS IN VECTOR NOTATION.

A vector is simply a sequence of ordinary numbers (scalars) arranged in a definite order:

$$\mathfrak{z} = (z_1, z_2, z_3, \quad).$$

The sum of two vectors

$$\mathfrak{y} = (y_1, y_2, y_3, \cdot \quad), \quad \mathfrak{z} = (z_1, z_2, z_3, \quad)$$

is defined as the vector obtained by adding their respective components:

$$\mathfrak{y} + \mathfrak{z} = (y + z_1, y_2 + z_2, y_3 + z_3, \quad).$$

The product of and $\mathfrak{z}$ is defined as the scalar

$$\mathfrak{y}\mathfrak{z} = y z_1 + y_2 z_2 + y_3 z_3 + - \quad .$$

If we have a sequence of vectors α_i $(i = 1, 2, - \quad)$, we lay down the definition:

$$(\alpha_1, \alpha_2, \alpha_3, \quad)\mathfrak{z} = (\alpha_1 \mathfrak{z}, \alpha_2 \mathfrak{z}, \alpha_3 \mathfrak{z}, \quad),$$

which is an ordinary vector.

Our system of equations

$$(1+a_{11})x_1 + a_{12}x_2 + a_{13}x_3 + \qquad = c_1,$$
$$a_{21}x_1 + (1+a_{22})x_2 + a_{23}x_3 + \qquad = c_2,$$
$$a_{31}x_1 + a_{32}x_2 + (1+a_{33})x_3 + \qquad = c_3,$$

may now be written

$$x_k = c_k - \alpha_k \xi \qquad (k = 1, 2, \qquad)$$

and abridged into the single equation

$$\xi = \gamma - (\alpha_1, \alpha_2, \alpha_3, \qquad)\xi,$$

where

$$\alpha_k = (a_{k1}, a_{k2}, a_{k3}, \qquad),$$
$$\gamma = (c_1, c_2, c_3, \qquad),$$
$$\xi = (x_1, x_2, x_3, \qquad).$$

Our approximation process

$$x_k^{(1)} = c_k,$$
$$x_k^{(i+1)} = c_k - \left[a_{k1}x_1^{(i)} + a_{k2}x_2^{(i)} + \ \right],$$

becomes in vector notation

$$\xi^{(1)} = \gamma,$$
$$\xi^{(i+1)} = \gamma - (\alpha_1, \alpha_2, \alpha_3, \qquad)\xi^{(i)},$$

Place $\xi = \xi^{(1)} + (\xi^{(2)} - \xi^{(1)}) + (\xi^{(3)}\ \xi^{(2)}) +$

Then

$$\xi^{(1)} = \gamma,$$

$$\xi^{(2)} - \xi^{(1)} = -(\alpha_1, \alpha_2, \alpha_3, \quad)\xi^{(1)},$$

$$\xi^{(3)} - \xi^{(2)} = -(\alpha_1, \alpha_2, \alpha_3, \quad)(\xi^{(2)} - \xi^{(1)}),$$

$$\xi^{(4)} - \xi^{(3)} = -(\alpha_1, \alpha_2, \alpha_3, \quad)(\xi^{(3)} - \xi^{(2)}),$$

$$\cdot$$

which by addition gives

$$\xi = \gamma - (\alpha_1, \alpha_2, \alpha_3, \quad)\xi,$$

and thus furnishes ξ as a a solution of the given equations.

Here, of course, our work has been merely formal, the justification of it in certain cases having been given previously.

9. SLIGHT CHANGES IN THE METHOD OF APPROXIMATION.

There are a number of variations on the method of approximation previously given, from which we get correspondingly different theorems. For example, we shall prove

Theorem VI. If the system

$$(22)\begin{cases}(1+a_{11})x_1 + a_{12}x_2 + a_{13}x_3 + \cdots & = c_1, \\ a_{21}x_1 + (1+a_{22})x_2 + a_{23}x_3 + \cdots & = c_2, \\ a_{31}x + a_{32}x + (1+a_{33})x_3 + \cdots & = c_3, \\ - \quad - & , \end{cases}$$

is restricted so that

$$|c_k| \leqq C, \quad \sum_{j=1}^{\infty} |a_{ij}| \leqq P \quad (i = 1, 2, \cdots), \text{ and}$$

$$\prod_{i=2}^{\infty} (1 + Q_i) < \frac{1}{p}, \text{ where } Q_i = \sum_{j=1}^{i-1} |a_{ij}|,$$

then (22) has one solution and only one solution such that the x_k are bounded.

To establish this theorem we modify the original approximation process by taking for the initial set of values:

$$\lambda_1^{(1)} = c_1 ,$$

$$\lambda_2^{(1)} = c_2 - \alpha_{21}\lambda_1^{(1)} ,$$

$$\lambda_3^{(1)} = c_3 - \alpha_{31}\lambda_1^{(1)} - \alpha_{32}\lambda_2^{(1)} ,$$

$$.$$

Thereafter we set

$$\lambda_k^{(i)} = c_k - [\alpha_{k1}\lambda_1^{(i)} + \alpha_{k2}\lambda_2^{(i)} + \quad + \alpha_{kk-1}\lambda_{k-1}^{(i)}$$
$$+ \alpha_{kk}\lambda_k^{(i-1)} + \alpha_{kk+1}\lambda_{k+1}^{(i-1)} + \quad].$$

From these equations we find

$$\lambda_k^{(1)} = c_k - [\alpha_{k1}\lambda_1^{(1)} + \alpha_{k2}\lambda_2^{(1)} + \quad + \alpha_{kk-1}\lambda_{k-1}^{(1)}] ,$$

(23)

$$\lambda_k^{(2)} - \lambda_k^{(1)} = -[\alpha_{k1}(\lambda_1^{(2)} - \lambda_1^{(1)}) + \quad + \alpha_{k,k-1}(\lambda_{k-1}^{(2)} - \lambda_k^{(1)})$$
$$+ \alpha_{kk}\lambda_k^{(1)} + \alpha_{k,k+1}\lambda_{k+1}^{(1)} + \quad] ,$$

$$\lambda_k^{(3)} - \lambda_k^{(2)} = -[\alpha_{k1}(\lambda_1^{(3)} - \lambda_1^{(2)}) + \quad + \alpha_{k,k-1}(\lambda_{k-1}^{(3)} - \lambda_{k-1}^{(2)})$$
$$+ \alpha_{kk}(\lambda_k^{(2)} - \lambda_k^{(1)}) + \alpha_{k,k+1}(\lambda_{k+1}^{(2)} - \lambda_{k+1}^{(1)}) + \quad ,$$

$$.$$

Hence $|\lambda_1| \leqq C ,$

$$|\lambda_2| \leqq C(1+\alpha_2) ,$$

$$|\lambda_3| \leqq C(1+\alpha_2)(1+\alpha_3) ,$$

$$|\lambda_k^{(1)}| \leqq C \cdot \prod_{i=2}^{k}(1+\alpha_i) \leqq C \cdot \prod_{i=2}(1+\alpha_i) ,$$

$$|x_1^{(2)} - x_1^{(1)}| \leqq \rho\cdot C\cdot \Pi,$$

$$|x_2^{(2)} - x_2^{(1)}| \leqq \rho\cdot C\cdot \Pi\cdot(1+Q_2),$$

$$|x_3^{(2)} - x_3^{(1)}| \leqq \rho\cdot C\cdot \Pi\cdot(1+Q_2)(1+Q_3),$$

$$|x_k^{(2)} - x_k^{(1)}| \leqq \rho\cdot C\cdot \Pi^2,$$

$$|x_k^{(3)} - x_k^{(2)}| \leqq \rho^2 C\cdot \Pi^3,$$

Hence we have

$$(24) \quad x_k = x_k^{(1)} + (x_k^{(2)} - x_k^{(1)}) + (x_k^{(3)} - x_k^{(2)}) + \cdots$$
$$\ll C\Pi + \rho C\Pi^2 + \rho C\Pi^3 + \cdots$$
$$= \frac{C\Pi}{1-\rho\Pi}.$$

The x_k as just defined surely form a solution of the original equations. For if we add the equations of (23) and sum by columns the absolutely convergent double series which results, we obtain

$$x_k = c_k - [a_{k1}x_1 + a_{k2}x_2 + \cdots].$$

From (24) we see that the x_k are bounded. Next we show that the x_k are the only solution of (22) for which this is true. For if x_k' and x_k'' are

two such solutions of (22), $\bar{X}_k = X'_k - X_k$ is a bounded solution of the homogeneous system corresponding to (22):

$$
(25) \qquad
\begin{aligned}
&(1+a_{11})\bar{X}_1 + a_{12}\bar{X}_2 + a_{13}\bar{X}_3 + \cdots = 0, \\
&a_{21}\bar{X}_1 + (1+a_{22})\bar{X}_2 + a_{23}\bar{X}_3 + \cdots = 0, \qquad |\bar{X}_k| \leq \overline{\underline{X}}, \\
&a_{31}\bar{X}_1 + a_{32}\bar{X}_2 + (1+a_{33})\bar{X}_3 + \cdots = 0, \qquad k = 1, 2, \ldots \\
&\cdots
\end{aligned}
$$

By carrying through our approximation process:

$$
\begin{aligned}
X_k^{(1)} &= \bar{X}_k, \\
X_k^{(i)} &= -\big[a_{k1}X_1^{(i)} + \cdots + a_{k,k-1}X_{k-1}^{(i)} \\
&\qquad + a_{kk}X_k^{(i-1)} + a_{k,k+1}X_{k+1}^{(i-1)} + \cdots\big],
\end{aligned}
$$

we get

$$
\begin{aligned}
|X_k^{(1)}| &\leq \overline{\underline{X}}, \\
|X_k^{(2)}| &\leq p\,\overline{\underline{X}}\,\Pi, \\
&\cdots \\
|X_k^{(i)}| &\leq p^{i-1}\,\overline{\underline{X}}\,\Pi^{i-1}.
\end{aligned}
$$

Then $\lim_{i=\infty} |X_k^{(i)}| = 0$, since $p\Pi < 1$. But we find from equations (25) that

$$\bar{X}_k = X_k^{(i)} \qquad (i = 1, 2, \ldots)$$

and therefore $\bar{X}_k = 0$, as we were to prove.

The last theorem will not be true if we

replace $\Pi < \frac{1}{\rho}$ by the restriction $\Pi \leq \frac{1}{\rho}$. For take

$$
\begin{aligned}
x_1 + \tfrac{1}{2} x_2 \qquad &= c_1, \\
x_1 + (1 - \tfrac{1}{2}) x_2 \qquad &= c_2, \\
x_3 \qquad &= c_3, \\
\cdots \qquad &\cdots ,
\end{aligned}
$$

$\Pi = 2$, $\rho = \frac{1}{2}$.

Clearly there are no solutions or an infinite number of solutions according as $c_1 \neq c_2$ or $c_1 = c_2$.

Next we take up another variation of the approximation method[1]. The theorem we shall prove by its use includes the existential part of von Koch's Theorem Ia.

Theorem VII. Suppose that (22) satisfies the hypotheses

1) $|c_k| \leqq C$, for $k = 1, 2, \cdots$,

2) $\sum_{j=1}^{\infty} |a_{k,N+j}| \leqq Q$ for $k = 1, 2, \cdots N$,

---

[1] The method here used was suggested to me by Professor Van Vleck.

3) $\sum_{j=1}^{N} |a_{N+k,j}| \leq A$ for $k = 1, 2,$,

4) $\sum_{j=1}^{\infty} |a_{N+k,N+j}| \leq \rho$ for $k = 1, 2,$,

5)
$$\Delta^{(N)} = \begin{vmatrix} 1+a_{11} & a_{12} & a_{1N} \\ & & \\ a_{N1} & a_{N2} & 1+a_{NN} \end{vmatrix} \neq 0 ,$$

$$M = \frac{\prod_{i=1}^{N} \left(1+\sum_{j=1}^{N} |a_{ji}|\right)}{\Delta^{(N)}} ,$$

and 6) $\rho + MAQ < 1$. Then the system (22) has one solution and only one solution for which the x_k are bounded.

The approximation process we use in the proof of this theorem is as follows: by means of the original process, find $x^{(1)}_{N+k}$ $(k=1, 2,$ $)$ to satisfy the system

$$(1+a_{N+1\,N+1})X^{(1)}_{N+1} + a_{N+1,N+2}X^{(1)}_{N+2} + \qquad = c_{N+1},$$

$$a_{N+2,N+1}X^{(1)}_{N+1} + (1+a_{N+2,N+2})X_{N+2} + \qquad = c_{N+2},$$

The possibility of this solution was shown in Theorem II. We substitute the values thus found into the first N equations of (22):

$$(1+a_{11})X^{(2)}_1 + a_{12}X^{(2)}_2 + \qquad + a_{1N}X^{(2)}_N = c_1 - a_{1N+1}X^{(1)}_{N+1} - \qquad ,$$

$$a_{N1}X^{(2)}_1 + a_{N2}X^{2}_2 + \qquad + (1+a_{NN}X^{(2)}_N) = c_N - a_{N\,N+1}X^{(1)}_{N+1} - \qquad ,$$

and solve these equations for $X^{(2)}_k$, $k=1,2,$ N. Next, return to the equations after the N-th, and determine $X^{(2)}_{N+k}$ $k=1,2,$, to satisfy the system

$$(1+a_{N+1,N+1})X^{(2)}_{N+1} + a_{N+1,N+2}X^{(2)}_{N+2} + \qquad = c_{N+1} - a_{N+1,1}X^{(2)}_1 - \dots - a_{N+1,N}X^{(2)}_N,$$

$$a_{N+2,N+1}X^{(2)}_{N+1} + (1+a_{N+2,N+2})X^{(2)}_{N+2} + \qquad = c_{N+2} - a_{N+2,1}X^{(2)}_1 - \qquad - a_{N+2,N}X^{(2)}_N,$$

,

which solution is always possible. Substitute the values $X^{(2)}_{N+k}$ $(k=1,2,\ \ )$ just found into the first N of the original equations (22) and solve these for $X^{(2)}_k$, $(k=1\ 2,\ \ N)$. The process thus

indicated is to be continued, working back and forth between the first N equations of system (22) and the system formed from (22) by omitting the first N equations.

We shall now demonstrate the convergence of the process given. It follows from Theorem II that

$$|\lambda^{(1)}_{N+k}| \leq \frac{C}{1-\rho},$$ and hence we have

$k = 1, 2,$,
$i = 1, 2, \ldots N.$

$$|\lambda^{(2)}_i| \leq M\left(C + \frac{QC}{1-\rho}\right),$$

$$|\lambda^{(2)}_{N+k} - \lambda^{(1)}_{N+k}| \leq \frac{AM}{1-\rho}\left(C + \frac{QC}{1-\rho}\right),$$

$$|\lambda^{(3)}_i - \lambda^{(2)}_i| \leq \frac{AM^2Q}{1-\rho}\left(C + \frac{QC}{1-\rho}\right),$$

$$|\lambda^{(3)}_{N+k} - \lambda^{(2)}_{N+k}| \leq \frac{A^2M^2Q}{(1-\rho)^2}\left(C + \frac{QC}{1-\rho}\right),$$

The series $\lambda_i = \lambda^{(1)}_i + (\lambda^{(2)}_i - \lambda^{(1)}_i) +$

and $\lambda_{N+k} = \lambda^{(1)}_{N+k} + (\lambda^{(2)}_{N+k} - \lambda^{(1)}_{N+k}) +$

converge, since $\frac{AMQ}{1-\rho} < 1$. It also follows that the λ_j are bounded.

The x_j as just defined actually are a solution of the system (22), as can be shown by the procedure we have followed so often before. Next we shall prove that there is but one solution for which the x_j are bounded. If we suppose the contrary, there would be a bounded solution $\bar{x}_k$ (whose bound is denoted by $\bar{X}$) of the homogeneous system corresponding to (22). Then we can find from the equations themselves that

$$|\bar{x}_{N+k}| \leqq A\bar{X} + \rho\bar{X}, \qquad k = 1, 2, \quad ,$$

$$|\bar{x}_i| \leqq MQ\bar{X}[A+\rho], \qquad i = 1, 2, \quad N.$$

$$|\bar{x}_{N+k}| \leqq \bar{X}[A+\rho][\rho + AMQ],$$

$$|\bar{x}_i| \leqq MQ\bar{X}[A+\rho][\rho + AMQ],$$

Hence $\bar{x}_j = 0$, $j = 1, 2, \quad .$

It is readily seen that the existential part of Theorem Ia is included in Theorem VII. Choose N so large that $\Delta^{N)} \neq 0$. When N is increased,

M cannot increase indefinitely. And simply by taking N sufficiently large we can make P, A, and Q as small as desired. Hence we can surely make $P + MAQ < 1$, which is all that remains to be done to prove the existential part of Theorem I a.

It is a corollary of this work that if $\Delta = 0$ in the case considered by von Koch (cf. Theorem I b), then for some fixed number N we must have $\Delta^{(n)} = 0$ for every $n > N$. Otherwise we could find an N large enough to satisfy all the restrictions of Theorem VII, and therefore the conclusion of Theorem VII would hold. This contradicts the conclusion of Theorem I b.

BIBLIOGRAPHY.

For a general exposition of the theory of von Koch, the reader is referred to F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues*, where a full treatment, with references, is given. See also von Koch, Sur les systèmes d'une infinité d'équations linéaires à une infinité d'inconnues, Compte Rendu du Congrès des Mathematiciens, Stockholm, 1909.

When this thesis was almost completely written, it was found that THEOREM II had been previously proved by von Koch, *Jahresbericht*, 1913, p.289. He used essentially the same method of sucessive approximation used in this thesis. However, he seems to have used this method to prove none of the other theorems proved in this thesis. THEOREMS III, IV, V, VI, and VII are new.

A comparison of the various methods of solving linear equations in infinitely many variables as well as some numerical examples are given by E. Goldschmidt, Würzburg Dissertation, 1912.

APPROVED, E. B. Van Vleck

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